On Some Metrics Compatible with the Fell-Matheron Topology

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Abstract

Modifying the Hausdorff-Buseman metric, we obtain a compatible metric with the Fell-Matheron topology on the space of closed subsets of a locally compact Hausdorff second countable space. We also give an alternative expression of this metric, and two more compatible metrics, a metric of summation form and a modified Rockafellar-Wets metric. Through the study of the relation between the original Hausdorff-Buseman metric and the Rockafellar-Wets metric, it is also shown that the convergence in the original Hausdorff-Buseman metric in the space of non-empty closed subsets of \( \mathbb{R}^N \) is equivalent to the Painlevé-Kuratowski convergence.

Key words: Fell-Matheron topology, Hausdorff-Buseman metric, Rockafellar-Wets metric, Painlevé-Kuratowski convergence

1. Introduction

The topologies on the space of closed sets has been studied extensively in connection with the theory and application to random sets, random fuzzy sets, statistics for coarse data, variational analysis, convex analysis, etc. Among them, the Fell-Matheron topology (or hit and miss topology) on the space of closed subsets (including the empty set) of a LCHS (locally compact Hausdorff second countable) space is one of the most basic and important topologies. Since the space of closed sets equipped with the Fell-Matheron topology is a separable compact Hausdorff space, it is metrizable. The Hausdorff-Buseman metric is introduced in Molchanov [3] as a concrete compatible metric. However, Wei and Wang [9] pointed out it does not describe the neighborhood of \( \emptyset \) in the Fell-Matheron topology (see also [5]). They also gave another compatible concrete metric which covers the defect above (see Section 2 below).

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Incidentally, Rockafellar and Wets [7] give a metric on the space of non-empty closed subsets of $\mathbb{R}^N$, and showed that the convergence in this metric is equivalent to the Painlevé-Kuratowski convergence in the space of non-empty closed subsets of $\mathbb{R}^N$.

The object of this paper is to give some simple compatible metrics with the Fell-Matheron topology on the space of closed subsets of a LCHS space. The first metrics we give are modified Hausdorff-Buseman metrics. To give this, we first show that there is a metric on the underlying LCHS space such that every bounded closed subsets is compact, and then use it for the exponent in the formula of Hausdorff-Buseman metric whereas we use truncated ones for the linear part in the formula (see (2.1)-(2.3) below). The key of our argument is Lemma 2.4 below, which estimates the truncated metric with the values close to the Hausdorff metric. We next give an alternative expression for this metric in (3.1)-(3.2), and another compatible metric of summation form in (3.3). We also give modified Rockafellar-Wets metrics in (3.7), which is compatible with the Fell-Matheron topology on the space of closed subsets of a LCHS space.

As a byproduct, we show that the original Hausdorff-Buseman metric is equivalent in some sense to the original Rockafellar-Wets metric. This implies that the convergence in the Hausdorff-Buseman metric on the space of non-empty closed subsets of $\mathbb{R}^N$ is equivalent to the Painlevé-Kuratowski convergence.

In the next Section 2, we define modified Hausdorff-Buseman metrics, and show that it is compatible with the Fell-Matheron topology. In Section 3, we give an alternative expression of this metric, and two more metrics, a metric of summation form and modified Rockafellar-Wets metrics. They are equivalent in some sense to the modified Hausdorff-Buseman metrics, and hence are compatible with the Fell-Matheron topology. Finally we show that the convergence in the Hausdorff-Buseman metric in the space of non-empty closed subsets of $\mathbb{R}^N$ is equivalent to the Painlevé-Kuratowski convergence.

2. Modified Hausdorff-Buseman metrics

Let $E$ be a LCHS (locally compact Hausdorff second countable) space, which is metrizable. We denote a compatible metric by $\rho'$. Let

| $\mathcal{F}(E)$ | the space of all closed subsets of $E$, |
| $\mathcal{F}'(E)$ | the space of all non-empty closed subsets of $E$, |
| $\mathcal{K}(E)$ | the space of all compact subsets of $E$, |
| $\mathcal{O}(E)$ | the space of all open subsets of $E$. |
For a family $\mathcal{H}$, denote $\mathcal{H}^F = \{A \in \mathcal{H} : A \cap F = \emptyset\}$ and $\mathcal{H}_G = \{A \in \mathcal{H} : A \cap G \neq \emptyset\}$.

**Definition 2.1.** The Fell-Matheron topology $\tau_f(E)$ on $\mathcal{F}(E)$ has a sub-base $\mathcal{F}(E)_G$ for all $G \in \mathcal{G}(E)$ and $\mathcal{F}(E)^K$ for all $K \in \mathcal{K}(E)$.

The space $\mathcal{F}(E)$ of closed subsets with Fell-Matheron topology is a separable compact Hausdorff space, so that it is metrizable. Hence the space $\mathcal{F}'(E)$ is also metrizable but is not closed in $\mathcal{F}(E)$ unless $E$ is compact.

Wei and Wang [9] gave a concrete metric in the case when $E$ is non-compact (see also [6]). Indeed, they first embedded $\mathcal{F}(E)$ into $\mathcal{F}'(E_\partial)$, where $E_\partial$ is the Alexandroff compactification of $E$, and then showed that the Hausdorff metric on $\mathcal{F}'(E_\partial)$ induces the Fell-Matheron topology on $\mathcal{F}(E)$.

The Hausdorff-Buseman metric is originally given by

$$d_{HB}(A, B) = \sup_{x \in E} e^{-\rho'(0,x)}|\rho'(x, A) - \rho'(x, B)|,$$

where $\rho'(x, A) = \inf\{\rho'(x, y) : y \in A\}$.

We will show that the metric space $(E, \rho)$ satisfies condition $\star$.

**Proof.** Since $E$ is a LCHS space, there is a compatible metric $\rho'$ on $E$ (see [1,9]). If $\rho'$ already satisfies condition $\star$, then we just set $\rho = \rho'$. If $\rho'$ does not satisfy condition $\star$, then the space $E$ is non-compact, because every closed set in a compact metric space is compact. Let $E_\partial$ be the Alexandroff compactification of $E$. Since $E_\partial$ is a separable compact Hausdorff space, there is a compatible metric $\tilde{\rho}$ on $E_\partial$. The diameter of $E_\partial$ measured with $\tilde{\rho}$ is finite. Let

$$\rho(x, y) = \tilde{\rho}(x, y) + |\tilde{\rho}(x, \partial)^{-1} - \tilde{\rho}(y, \partial)^{-1}|, \quad x, y \in E.$$

We will show that the metric space $(E, \rho)$ satisfies condition $\star$. Take a bounded closed set $D$ in $(E, \rho)$. Then, there is a $p \in E$ and $M > 0$ such that $\rho(p, x) \leq M$, $x \in D$. It then follows that $\tilde{\rho}(x, \partial)^{-1} \leq M + \tilde{\rho}(p, \partial)^{-1}$, $x \in D$, which implies $\tilde{\rho}(\partial, D) > 0$. Since $\rho$ and $\tilde{\rho}$ induce the same topology.

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1 We use the convention $\inf\emptyset = \infty$ and $\sup\emptyset = 0$.

2 A metric space $X$ equipped with the metric $\rho$ is denoted simply by $(X, \rho)$. 

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on $E$, the set $D$ is closed in $(E_0, \tilde{\rho})$ and so is compact there, because $E_0$ is compact. Due to $\tilde{\rho}(\partial, D) > 0$, this proves that $D$ is also compact in $(E, \rho)$. \qed

In the sequel, we assume that the metric $\rho$ always satisfies condition $(\ast)$ and $p$ is a fixed point in $E$. For each $\theta, b > 0$, let

$$d^{\theta,b}(A, B) = \sup_{x \in E} e^{-\theta \rho(p,x)} |\rho(x, A) \land b - \rho(x, B) \land b|, \quad (2.1)$$

$$d^\theta(A, B) = \sup_{x \in E} e^{-\theta \rho(p,x)} |\rho(x, A) - \rho(x, B)|, \quad (2.2)$$

where $\rho(x, A) = \inf\{\rho(x, y) : y \in A\}$. We call them modified Hausdorff-Buseman metrics, because the original Hausdorff-Buseman metric $d_{HB}$ is just $d^1$ with $p = 0$. Note that the convention implies $\rho(x, \emptyset) = \infty$ for all $x \in E$ and $d^{\theta,b}(\emptyset, \emptyset) = 0$. The parameters $\theta$ and $b$ are only for comparison with other metrics, and do not change the basic feature in the following arguments. The simplest case is of course $\theta = b = 1$, that is

$$d(A, B) := d^{1,1}(A, B) = \sup_{x \in E} e^{-\rho(p,x)} |\rho(x, A) \land 1 - \rho(x, B) \land 1|. \quad (2.3)$$

We denote $D_r := \{x \in E : \rho(p, x) \leq r\}$, the closed ball with radius $r > 0$ with the center at $p$. By convention, we set $D_0 = \emptyset$.

**Lemma 2.3.** For each $\theta, b > 0$, $d^{\theta,b}$ is a metric on $\mathcal{F}(E)$.

**Proof.** Clearly, the only thing we have to show is that $d^{\theta,b}(A, B) = 0$ implies $A = B$. Suppose that $A \neq B$. Without loss of generality, we may assume $A \setminus B \neq \emptyset$, which implies the existence of an $x_0 \in A$ with $x_0 \notin B$. We then have $\rho(x_0, A) \land b = 0$ and $\rho(x_0, B) \land b > 0$ (because $B$ is a closed set). This yields $d^{\theta,b}(A, B) \geq e^{-\theta \rho(p,x_0)} \rho(x_0, B) \land b > 0$. \qed

For $b > 0$, let

$$d^\theta_H(A, B) = \sup_{x \in A} \rho(x, B), \quad d^b_H(A, B) = \sup_{x \in A} \rho(x, B) \land b.$$ 

From the convention, $d^b_H(\emptyset, A) = 0$ for $A \in \mathcal{F}(E)$ and $d^b_H(A, \emptyset) = b$ for $A \in \mathcal{F}(E)$. We note that $d^b_H(A, B) \lor d^b_H(B, A) = d_H(A, B) \land b$, where $d_H(A, B) = d^\theta_H(A, B) \lor d^b_H(A, B)$, the Hausdorff distance between $A$ and $B$.

**Lemma 2.4.** Let $\theta, b > 0$. Then, for each $r > 0$, it holds

$$d^b_H(A \cap D_r, B) \lor d^b_H(B \cap D_r, A) \leq \sup_{x \in D_r} |\rho(x, A) \land b - \rho(x, B) \land b| \quad (2.4)$$

$$\leq d^b_H(A \cap D_{r+b}, B) \lor d^b_H(B \cap D_{r+b}, A).$$

\[3\] $a \lor b$ and $a \land b$ stand for the maximum and minimum of $a$ and $b$, respectively.
Proof. To show the first inequality in (2.4), we first note that

\[ d^b_H(A \cap D_r, B) = \sup_{x \in A \cap D_r} \rho(x, B) \wedge b \leq \sup_{x \in D_r} |\rho(x, A) \wedge b - \rho(x, B) \wedge b|. \]

Indeed, if \( A \cap D_r = \emptyset \), this is obvious by the convention, whereas, in the case where \( A \cap D_r \neq \emptyset \), this also holds because \( \rho(x, A) = 0 \) for all \( x \in A \cap D_r \). Similarly, we have

\[ d^b_H(B \cap D_r, A) = \sup_{x \in B \cap D_r} \rho(x, A) \wedge b \leq \sup_{x \in D_r} |\rho(x, A) \wedge b - \rho(x, B) \wedge b|. \]

Combining these two relations, we obtain the first inequality in (2.4).

To show the second inequality in (2.4), we will first show that

\[ |\rho(x, A) \wedge b - \rho(y, A) \wedge b| \leq \rho(x, y) \wedge b, \quad x, y \in E, \ A \in \mathcal{F}(E). \quad (2.5) \]

Since (2.5) is clear for \( A = \emptyset \), we assume \( A \neq \emptyset \). Then, for each \( z \in A \), it holds

\[ \rho(x, A) \wedge b \leq \rho(x, z) \wedge b \leq \rho(x, y) \wedge b + \rho(y, z) \wedge b. \]

Taking the infimum over \( z \in A \), we have \( \rho(x, A) \wedge b \leq \rho(x, y) \wedge b + \rho(y, A) \wedge b \). Exchanging the roles of \( x \) and \( y \), we also have \( \rho(y, A) \wedge b \leq \rho(x, y) \wedge b + \rho(x, A) \wedge b \). These two relations imply (2.5).

We will next show that

\[ \rho(x, B) \wedge b - \rho(x, A) \wedge b \leq d^b_H(A \cap D_{r+b}, B), \quad x \in D_r. \quad (2.6) \]

Let \( x \in D_r \). If \( \rho(x, A) \geq b \), then the left hand side in (2.6) is nonpositive, and (2.6) holds. We thus assume that \( \rho(x, A) < b \). Then for each \( 0 < \varepsilon < b - \rho(x, A) \), there is a \( y \in A \) such that \( \rho(x, y) < \rho(x, A) + \varepsilon < b \). This implies \( y \in D_{r+b} \) and so \( y \in A \cap D_{r+b} \). Moreover,

\[ \rho(x, B) \wedge b - \rho(x, A) \wedge b \leq \rho(x, B) \wedge b - \rho(x, y) \wedge b + \varepsilon. \]

By (2.5) with \( A \) replaced by \( B \), we have \( \rho(x, B) \wedge b - \rho(x, y) \wedge b \leq \rho(y, B) \wedge b \), and so

\[ \rho(x, B) \wedge b - \rho(x, A) \wedge b \leq \rho(y, B) \wedge b + \varepsilon \leq d^b_H(A \cap D_{r+b}, B) + \varepsilon. \]
Since \( \varepsilon \) can be taken arbitrarily close to 0, we obtain (2.6). Exchanging the roles of \( A \) and \( B \), we have

\[
\rho(x, A) \land b - \rho(x, B) \land b \leq d_H^b(B \cap D_{r+b}, A), \quad x \in D_r,
\]

which with (2.6) proves the second inequality in (2.4). \( \square \)

**Theorem 2.5.** Let \( E \) be a LCHS space and \( \theta, b > 0 \). Then the topology \( \tau_{d,b} \) on \( \mathcal{F}(E) \) induced by the metric \( d^{p,b} \) coincides with the Fell-Matheron topology \( \tau_f(E) \) on \( \mathcal{F}(E) \).

**Proof.** We will first show that \( \tau_{d,b} \) is coarser than \( \tau_f(E) \).

Take an \( A \in \mathcal{F}(E) \) and an \( \varepsilon \in (0, 1) \). Choose then an \( r > 0 \) such that \( b \exp(-\theta r) < \varepsilon \), or \( \sup_{x \notin D_r} \exp(-\theta p(x)) \left| \rho(x, A) \land b - \rho(x, B) \land b \right| < \varepsilon \). Since \( A \cap D_{r+b} \) is compact, there is a finite open covering \( G_1, G_2, \ldots, G_n \) of \( A \cap D_{r+b} \) whose diameters are less than \( \varepsilon/2 \) and \( G_i \cap A \neq \emptyset \) for every \( i = 1, 2, \ldots, n \). Let \( K = D_{r+b} \setminus (\bigcup_{i=1}^n G_i) \), which is a compact set, and take a \( B \in \mathcal{F}(E)^K \cap (\cap_{i=1}^n \mathcal{F}(E)_{G_i}) \).

We will show

\[
d_H^b(A \cap D_{r+b}, B) \lor d_H^b(B \cap D_{r+b}, A) \leq \varepsilon/2. \tag{2.7}
\]

Indeed, for each \( x \in A \cap D_{r+b} \), there is an \( i \) such that \( x \in G_i \). Since \( G_i \cap B \neq \emptyset \) and the diameter of \( G_i \) is less than \( \varepsilon/2 \), we have \( \rho(x, B) < \varepsilon/2 \), whence \( d_H^b(A \cap D_{r+b}, B) \leq \varepsilon/2 \) follows. On the other hand, \( B \in \mathcal{F}(E)^K \) implies \( B \cap D_{r+b} \subset D_{r+b} \setminus K \subset \bigcup_{i=1}^n G_i \). Thus every \( x \in B \cap D_{r+b} \) is included by a \( G_i \) for some \( i \). Since \( A \cap G_i \neq \emptyset \), we have \( \rho(x, A) < \varepsilon/2 \), so that \( d_H^b(B \cap D_{r+b}, A) \leq \varepsilon/2 \). This with the former formula \( d_H^b(A \cap D_{r+b}, B) \leq \varepsilon/2 \) proves (2.7).

Now (2.7) with (2.4) implies \( \sup_{x \in D_r} \left| \rho(x, A) \land b - \rho(x, B) \land b \right| \leq \varepsilon/2 \). Hence,

\[
\sup_{x \in E} \exp(-\theta p(x)) \left| \rho(x, A) \land b - \rho(x, B) \land b \right| \leq \sup_{x \in D_r} \exp(-\theta p(x)) \left| \rho(x, A) \land b - \rho(x, B) \land b \right| \\
\lor \sup_{x \notin D_r} \exp(-\theta p(x)) \left| \rho(x, A) \land b - \rho(x, B) \land b \right| < \varepsilon.
\]

This yields \( \mathcal{F}(E)^K \cap (\cap_{i=1}^n \mathcal{F}(E)_{G_i}) \subset \{ B \in \mathcal{F}(E) : d^{p,b}(A, B) < \varepsilon \} \), and \( \tau_{d,b} \) is coarser than \( \tau_f(E) \).

The assertion above ensures the identity map \( \iota \) on \( \mathcal{F}(E) \) is continuous from \( (\mathcal{F}(E), \tau_f(E)) \) to \( (\mathcal{F}(E), \tau_{d,b}) \).\(^4\) Further the topological sapce \( (\mathcal{F}(E), \tau_f(E)) \) is compact, and \( (\mathcal{F}(E), \tau_{d,b}) \) is a Hausdorff space, because it is induced by the metric \( d^{p,b} \). Hence \( \iota \) is a homomorphism by a well known theorem on topology (see [8] e.g.). This means that the two topologies \( \tau_f(E) \) and \( \tau_{d,b} \) are the same. \( \square \)

\(^4\) A topological space \( X \) equipped with the topology \( \tau \) is simply denoted by \( (X, \tau) \).
Remark 1. Suppose that $E$ is compact. Then the metric sapce $(E, \rho)$ is bounded, and so is included by a $D_{r_0}$ for some $r_0 > 0$. We then have

$$e^{-\theta r_0}d_H^b(A, B) \leq d_H^{\ast b}(A, B) \leq d_H^b(A, B), \quad (2.8)$$

where $d_H^b(A, B) = d_H^b(\emptyset, B) \vee d_H^b(A, B)$. Especially, $d_H^{\ast b}(\emptyset, A) \geq e^{-\theta r_0}d_H^b(\emptyset, A) = b e^{-\theta r_0}$ for $A \in \mathcal{F}(E)$. Thus $\emptyset$ is an isolated point and $\mathcal{F}(E)$ is compact (since $\mathcal{F}(E)$ is compact with respect to the Fell-Matheron topology).

Example 1. Let $E = \mathbb{R}^N$. Then the Euclidean metric $\rho_E$ already satisfies condition (\textdagger). Hence we have a modified Hausdorff-Buseman metric $d^{\ast b}$ on $\mathcal{F}(\mathbb{R}^N)$ in (2.1) for $\rho = \rho_E$ and $p = O$ the origin. Let $N = 1$ and $A_n = \{n\}$, $n \in \mathbb{N}$. Then $\rho_E(x, A_n) \wedge \rho_E(A_n, x) = 1$ for $|x - n| \geq 1$, and $\rho_E(x, A_n) = |x - n|$ for $|x - n| < 1$. Hence $d(A_n, \emptyset) = \sup_{|x - n| < 1} e^{-|x|}(1 - |x - n|) = e^{-n}$. Similarly, for $m, n \in \mathbb{N}$,

$$d(A_m, A_n) = \left(\sup_{|x - n| < 1} e^{-|x|}(1 - |x - m|)\right) \vee \left(\sup_{|x - n| < 1} e^{-|x|}(1 - |x - n|)\right) = e^{-(m \wedge n)},$$

which directly shows that $\{A_n\}$ is a Cauchy sequence in $(\mathcal{F}(\mathbb{R}^N), d)$.

Example 2. Let $S^N$ be the unit sphere with the center at $e_{N+1} = (0, \ldots, 0, 1)$ embedded in the Euclidean sapce $\mathbb{R}^{N+1}$. Then the geodesic metric $\rho_S$ on $S^N$ satisfies condition (\textdagger), because $S^N$ is compact with respect to the induced topology. The modified Hausdorff-Buseman metric $d^{\theta, \pi}$ on $\mathcal{F}(S^N)$ is defined through (2.1) for $\rho = \rho_S$ and $p = O$ the origin, which coincides with $d^\theta$ (because the diameter of $S^N$ measured by $\rho_S$ equals $\pi$). With respect to this metric, the empty set $\emptyset$ is an isolated point and $\mathcal{F}(S^N)$ is compact. By Theorem 2.5, the topology $\tau_{\theta, \pi}$ on $\mathcal{F}(S^N)$ induced by the metric $d^{\theta, \pi}$ coincides with the Fell-Matheron topology $\tau_{\mathcal{F}}(S^N)$ on $\mathcal{F}(S^N)$. Hence the relative topology $\tau_{d^{\theta, \pi}}$ on $\mathcal{F}'(S^N)$ of $\tau_{\theta, \pi}$ on $\mathcal{F}(S^N)$ coincides with the relative Fell-Matheron topology $\tau'_{\mathcal{F}}(S^N)$ on $\mathcal{F}'(S^N)$. We note that, from (2.8), the metric $d^{\theta, \pi}$ is equivalent to the Hausdorff metric $d_H$ with respect to the metric $\rho_S$ on $\mathcal{F}'(S^N)$.

Incidentally, there is a stereographic projection $\psi : \mathbb{R}^N \to S^N \setminus \{\partial\}$, where $\partial$ is the north pole $2e_{N+1}$ of $S^N$. This defines a one-to-one map $\Psi$ from $\mathcal{F}(\mathbb{R}^N)$ onto $\mathcal{F}'(S^N)$ through

$$\Psi(A) = \text{cl}\{\psi(x) : x \in A\}, \quad A \in \mathcal{F}'(\mathbb{R}^N), \quad \Psi(\emptyset) = \{\partial\},$$

where the closure is taken in $S^N$. Further, it is easily seen that $\Psi$ is a homeomorphism from $(\mathcal{F}(\mathbb{R}^N), \tau_{\mathcal{F}}(\mathbb{R}^N))$ onto $(\mathcal{F}'(S^N), \tau'_{\mathcal{F}}(S^N))$. Hence the metric defined by $d_{\mathbb{R}^N}(A, B) = d_{\mathbb{R}^N}^{\psi, \pi}(\Psi(A), \Psi(B)), A, B \in \mathcal{F}(\mathbb{R}^N)$ is compatible with the Fell-Matheron topology $\tau_{\mathcal{F}}(\mathbb{R}^N)$. This metric $d_{\mathbb{R}^N}$ is basically same as what [6] and [9] gave in this case.
3. Equivalent metrics

In this section, we give an alternative expression of the metric, and also discuss some other equivalent metrics to $d_{\theta,b}$.

**Proposition 3.1.** For each $\theta, b > 0$, it holds

$$d_{\theta,b}(A, B) = \tilde{d}_{\theta,b}(A, B), \quad A, B \in \mathcal{F}(E), \quad (3.1)$$

where

$$\tilde{d}_{\theta,b}(A, B) = \sup_{r > 0} e^{-\theta r} \sup_{x \in D_r} |\rho(x, A) \wedge b - \rho(x, B) \wedge b|. \quad (3.2)$$

**Proof.** Let $f(x) = |\rho(x, A) \wedge b - \rho(x, B) \wedge b|$. It then holds that

$$e^{-\theta r} \sup_{x \in D_r} f(x) \leq \sup_{x \in D_r} e^{-\theta \rho(p,x)} f(x) \leq d_{\theta,b}(A, B), \quad r > 0.$$

Hence, we have $\tilde{d}_{\theta,b}(A, B) \leq d_{\theta,b}(A, B)$.

For the reverse inequality, take an $\varepsilon > 0$, and then choose an $x_0 \in E$ such that $e^{-\theta \rho(p,x_0)} f(x_0) > d_{\theta,b}(A, B) - \varepsilon$. Denoting $r = \rho(p,x_0)$, we have

$$d_{\theta,b}(A, B) - \varepsilon < e^{-\theta \rho(p,x_0)} f(x_0) \leq e^{-\theta r} \sup_{x \in D_r} f(x) \leq \tilde{d}_{\theta,b}(A, B).$$

Since $\varepsilon > 0$ is arbitrary, we obtain $d_{\theta,b}(A, B) \leq \tilde{d}_{\theta,b}(A, B)$. □

For each $\theta, b > 0$, let

$$\hat{d}_{\theta,b}(A, B) = \sum_{k=1}^{\infty} e^{-k\theta} \sup_{x \in D_k} |\rho(x, A) \wedge b - \rho(x, B) \wedge b|, \quad A, B \in \mathcal{F}(E), \quad (3.3)$$

$$\hat{\hat{d}}_{\theta,b}(A, B) = \sum_{k=1}^{\infty} e^{-k\theta} (d_H^b(A \cap D_k, B) \vee d_H^b(B \cap D_k, A)) \quad A, B \in \mathcal{F}(E). \quad (3.4)$$

From (2.4), it follows

$$\hat{\hat{d}}_{\theta,b}(A, B) \leq \hat{d}_{\theta,b}(A, B) \leq e^{[b] \theta} \hat{d}_{\theta,b}(A, B) \quad A, B \in \mathcal{F}(E), \quad (3.5)$$

where $[b]$ is the smallest integer which is larger than or equal to $b$. Hence, the sub-base of open balls $\{B \in \mathcal{F}(E) : \hat{\hat{d}}_{\theta,b}(A, B) < \varepsilon\}$, for all $A \in \mathcal{F}(E), \varepsilon > 0$ and $\{B \in \mathcal{F}(E) : \hat{d}_{\theta,b}(A, B) < \varepsilon\}$, for all $A \in \mathcal{F}(E), \varepsilon > 0$ induce the same topology on $\mathcal{F}(E)$. The quantity $\hat{\hat{d}}_{\theta,b}(A, B)$ is easy to comput. But it does not satisfy the triangular inequality in general. We have
Theorem 3.2. For each $\theta$, $\theta'$, $b > 0$ with $\theta' < \theta$, it holds

$$e^{-\theta} d^{\theta,b}(A, B) \leq d^{\theta,b}(A, B) \leq (e^{\theta - \theta'} - 1)^{-1} d^{\theta',b}(A, B) \quad A, B \in \mathcal{F}(E). \quad (3.6)$$

Hence, the metric $\tilde{d}^{\theta,b}$ is compatible with the Fell-Matheron topology $\tau_f(E)$ on $\mathcal{F}(E)$.

Proof. Let $f(x) = |\rho(x, A) \wedge b - \rho(x, B) \wedge b|$. It then follows that

$$\sup_{x \in D_k \setminus D_{k-1}} e^{-\theta \rho(p,x)} f(x) \leq e^{-\theta(k-1)} \sup_{x \in D_k \setminus D_{k-1}} f(x) \leq e^\theta c^{-k\theta} \sup_{x \in D_k} f(x).$$

for all $k \in \mathbb{N}$. Hence

$$d^{\theta,b}(A, B) = \sup_{k \in \mathbb{N}} \sup_{x \in D_k \setminus D_{k-1}} e^{-\theta \rho(p,x)} f(x) \leq e^\theta \sup_{k \in \mathbb{N}} e^{-k\theta} \sup_{x \in D_k} f(x) \leq e^\theta d^{\theta,b}(A, B),$$

which proves the first inequality in (3.6).

We next note that

$$e^{-k\theta} \sup_{x \in D_k} f(x) \leq e^{-k(\theta - \theta')} \sup_{x \in D_k} e^{-\theta \rho(p,x)} f(x) \leq e^{-k(\theta - \theta')} d^{\theta,b}(A, B).$$

This implies $d^{\theta,b}(A, B) \leq d^{\theta,b}(A, B) \sum_{k=1}^{\infty} e^{-k(\theta - \theta')}$, which proves the second inequality in (3.6).

The latter assertion follows form (3.6) and Theorem 2.5. □

In [7], Rockafellar and Wets gave the metric

$$d_{RW}(A, B) = \int_{0}^{\infty} e^{-r} \sup_{x \in D_r} |\rho_{E}(x, A) - \rho_{E}(x, B)| dr, \quad A, B \in \mathcal{F}'(\mathbb{R}^N),$$

where $\rho_{E}$ is the Euclidean metric on $\mathbb{R}^N$. We modify it a little bit. For each $\theta$, $b > 0$, let

$$d_{RW}^{\theta,b}(A, B) = \int_{0}^{\infty} e^{-\theta r} \sup_{x \in D_r} |\rho(x, A) \wedge b - \rho(x, B) \wedge b| dr, \quad A, B \in \mathcal{F}(E),$$

which we call modified Rockafellar-Wets metrics.

Theorem 3.3. For each $\theta$, $\theta'$, $b > 0$ with $\theta' < \theta$, it holds

$$e^{-2\theta} d^{\theta,b}(A, B) \leq d_{RW}^{\theta,b}(A, B) \leq (\theta - \theta')^{-1} d^{\theta,b}(A, B) \quad A, B \in \mathcal{F}'(E). \quad (3.8)$$

Hence, the metric $d_{RW}^{\theta,b}$ is compatible with the Fell-Matheron topology $\tau_f(E)$ on $\mathcal{F}(E)$.
Proof. Let \( f(x) = |\rho(x, A) \land b - \rho(x, B) \land b| \). Since \( e^{-k\theta} \sup_{x \in D_k} f(x) \leq e^\theta \sup_{x \in D_r} f(x) \) for \( k \leq r \leq k + 1 \), it holds

\[
\bar{d}^{\theta,b}(A, B) \leq e^\theta \sum_{k=1}^{\infty} \int_k^{k+1} e^{-\theta r} \sup_{x \in D_r} f(x) \, dr \leq e^\theta d_{RW}^{\theta,b}(A, B).
\]

This with (3.6) proves the first inequality in (3.8).

On the one hand, it holds

\[
d^{\theta,b}_{RW}(A, B) = \int_0^\infty e^{-(\theta-\theta')r} e^{-\theta' r} \sup_{x \in D_r} f(x) \, dr.
\]

Since \( e^{-\theta' r} \sup_{x \in D_r} f(x) \leq \sup_{x \in D_r} e^{-\theta' \rho(x)} f(x) \), we obtain the second inequality in (3.8).

The latter assertion follows from (3.8) and Theorem 2.5. \( \Box \)

For each \( \theta > 0 \), let

\[
d^{\theta}_{RW}(A, B) = \int_0^\infty e^{-\theta r} \sup_{x \in D_r} |\rho(x, A) - \rho(x, B)| \, dr, \quad A, B \in \mathcal{F}'(E).
\]

By the same argument as that above, we can also show the following Corollary.

**Corollary 3.4.** For each \( \theta, \theta' > 0 \) with \( \theta' < \theta \), it holds

\[
e^{-2\theta} d^{\theta}(A, B) \leq d^{\theta}_{RW}(A, B) \leq (\theta - \theta')^{-1}d^{\theta'}(A, B), \quad A, B \in \mathcal{F}'(E). \tag{3.9}
\]

Let \( E = \mathbb{R}^N \), \( \rho = \rho_E \) and \( p = O \). For \( \theta > 1 \), we have from (3.9) that

\[
(\theta - 1)d^{\theta}_{RW}(A, B) \leq d_{HB}(A, B) \leq e^2 d_{RW}(A, B), \quad A, B \in \mathcal{F}'(\mathbb{R}^N), \tag{3.10}
\]

where \( d_{HB} = d^1 \) is the original Hausdorff-Buseman metric. The original Rockafellar-Wets metric \( d_{RW} \) is just in the case of \( \theta = 1 \). They showed that the convergence in the \( d_{RW} \) in \( \mathcal{F}'(\mathbb{R}^N) \) is equivalent to the Painlevé-Kuratowski convergence, that is for \( A, A_1, A_2, \ldots \in \mathcal{F}'(\mathbb{R}^N) \), it holds that \( \lim_n d_{RW}(A_n, A) = 0 \) if and only if

\[
A = \lim \inf_n A_n = \lim \sup_n A_n \tag{3.11}
\]

where

\[
\lim \inf_n A_n = \{ x = \lim_{n \to \infty} x_n : x_n \in A_n, \ n \in \mathbb{N} \},
\]

\[
\lim \sup_n A_n = \{ x = \lim_{m \to \infty} x_m : x_m \in A_m, \ m \in \mathbb{M}, \ \text{for some } \mathbb{M} \subset \mathbb{N} \}.
\]
Note that
\[ d_{RW}^\theta(A, B) = \theta^{-1} \int_0^\infty e^{-r} \sup_{x \in D_{r/\theta}} |\rho(x, A) - \rho(x, B)|dr, \]
and the integral in the right hand side is the Rockafellar-Wets metric with respect to the metric $\theta \rho_E$. Since $\theta \rho_E$ is equivalent to $\rho_E$, the convergence in $d_{RW}^\theta$ in $\mathcal{F}(\mathbb{R}^N)$ is equivalent to the convergence in $d_{RW}$. Hence, due to (3.10), we have following

**Corollary 3.5.** The convergence in the Hausdorff-Basean metric $d_{HB}$ in $\mathcal{F}(\mathbb{R}^N)$ is equivalent to the Painlevé-Kuratowski convergence (3.11).

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**References**


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